

On The Dynamics of a Certain Fourth Order Difference Equation with Constant Coefficients

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Abstract

This work investigates the relationship between difference equations and differential equations of the fourth order. In particular, we find the best discrete analogue of a certain differential equation. Solution behaviors of the difference and differential equations are compared.

Keywords: constant coefficients, difference equations, differential equations

I. Introduction

A fourth order difference equation, which is reported to be the "best" discrete analogue of a certain fourth order differential equation, is studied. We see why it is an excellent discrete analogue but we also see that the solutions of the difference equation have a much richer assortment of behaviors than the solutions of the corresponding differential equation.

Consider the differential equation

(e1) y'''' - y = 0.

The general solution of (e1) is

y = c1e^t + c2e^-t + c3 cos t + c4 sin t.

Generalizing (e1) slightly, consider the equation

(e2) y'''' - A^4 y = 0,

Where A is a positive constant.

The general solution of (e2) is

y = c1e^At + c2e^-At + c3 cos At + c4 sin At.

The solution y1 = e^At is called a strongly monotonic solution since y1 > 0, y1' > 0, y1'' > 0, y1''' > 0.

While the solution y2 = e^-At is termed a weakly monotonic solution because

y2 > 0, y2' < 0, y2'' > 0, y2''' < 0.

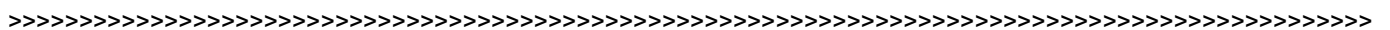
The solutions y3 = cos At and y4 = sin At are bounded oscillatory solutions with constant amplitudes. In other works, the solutions y1 and y2 have been referred to as strongly increasing functions and strongly decreasing functions, respectively.

A fourth order difference equation having the same characteristic equation as (e2) is

(e3) un+4 - A^4 un = 0, A > 0

Whose general solution is

un = c1A^n + c2(-A)^n + c3A^n cos npi/2 + c4A^n sin npi/2.



Clearly, the solutions of  $(e_3)$  behave very differently than those of  $(e_2)$ . Indeed, all solutions of  $(e_3)$  converge to 0 when  $A < 1$ , which is not possible for solutions of  $(e_2)$ ; all solutions of  $(e_3)$  are periodic when  $A = 1$ , another impossibility for solutions of  $(e_2)$ ; and finally, when  $A > 1$ , all nontrivial solutions of  $(e_3)$  are unbounded. Of course both  $(e_2)$  and  $(e_3)$  have oscillatory and nonoscillatory solutions. But because of the wide disparity in the behavior of solutions,  $(e_3)$  is not considered as a "good" discrete analogue of  $(e_2)$ .

Another possible discrete analogue of  $(e_2)$  is the difference equation

$$(e_4) \quad \Delta^4 u_n - A^4 u_n = 0, A > 0$$

Where  $\Delta$  denotes the forward difference operator, i.e.  $\Delta u_n = u_{n+1} - u_n$ . However, when  $A = 1$ , this equation is no longer of order 4. So  $(e_4)$  seems to fail as a "good" discrete analogue of  $(e_2)$ . A search of the literature, see [1], [3], leads one to believe that the equation

$$(1) \quad \Delta^4 u_n - A^2 u_{n+2} = 0, A > 0$$

Is probably the "best" discrete analogue of  $(e_2)$ . But why? This work attempts to answer this question. For additional studies that compare properties of solutions of differential equations to those of difference equations with constant coefficients, see [2].

$$\text{II. } (1) \quad \Delta^4 u_n - A^2 u_{n+2} = 0, A > 0$$

Since (1) has constant coefficients, to solve this equation requires the use of an associated characteristic equation. The characteristic equation of (1) is

$$(t-1)^4 - A^2 t^2 = 0 \text{ Or } [(t-1)^2 - At] \cdot [(t-1)^2 + At] = 0,$$

which leads to the following pair of quadratic equations

$$(2) \quad (t-1)^2 - At = 0$$

$$(3) \quad (t-1)^2 + At = 0.$$

The solutions of (2) are

$$t = \frac{2 + A \pm \sqrt{(2+A)^2 - 4}}{2} = \frac{2 + A \pm \sqrt{A^2 + 4A}}{2}.$$

$$\text{Let } t_1 = \frac{2 + A + \sqrt{A^2 + 4A}}{2} \text{ and } t_2 = \frac{2 + A - \sqrt{A^2 + 4A}}{2}.$$

Similarly, the solutions of (3) are

$$t = \frac{2 - A \pm \sqrt{(2-A)^2 - 4}}{2} = \frac{2 - A \pm \sqrt{A^2 - 4A}}{2}.$$

$$\text{Let } t_3 = \frac{2 - A + \sqrt{A^2 - 4A}}{2} \text{ and } t_4 = \frac{2 - A - \sqrt{A^2 - 4A}}{2}.$$

### III. Types of Solutions

Since  $A^2 + 4A > 0$ , the first two roots are real and determine, respectively, a strongly monotonic solution and a weakly monotonic solution. To see this:

Since  $t_1 = 1 + \frac{A + \sqrt{A^2 + 4A}}{2} > 1$ , the corresponding solution is strongly monotonic because  $u_n = t_1^n > 0, \Delta u_n = t_1^n(t_1 - 1) > 0, \Delta^2 u_n = t_1^n(t_1 - 1)^2 > 0, \Delta^3 u_n = t_1^n(t_1 - 1)^3 > 0.$

Since  $t_2 = 1 + \frac{A - \sqrt{A^2 + 4A}}{2}$ , we will show that  $0 < t_2 < 1$  and thus the corresponding solution is weakly monotonic because

$u_n = t_2^n > 0, \Delta u_n = t_2^n(t_2 - 1) < 0, \Delta^2 u_n = t_2^n(t_2 - 1)^2 > 0, \Delta^3 u_n = t_2^n(t_2 - 1)^3 < 0.$

To this end, consider the inequality

$$A^2 < A^2 + 4A < (A + 2)^2$$

which implies

$$A < \sqrt{A^2 + 4A} < A + 2.$$

So





