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On The Dynamics of a Certain Fourth Order Difference Equation with Constant Coefficients

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Abstract

This work investigates the relationship between difference equations and differential equations of the fourth order. In particular, we find the best discrete analogue of a certain differential equation. Solution behaviors of the difference and differential equations are compared.

Keywords: constant coefficients, difference equations, differential equations

I. Introduction

A fourth order difference equation, which is reported to be the "best" discrete analogue of a certain fourth order differential equation, is studied. We see why it is an excellent discrete analogue but we also see that the solutions of the difference equation have a much richer assortment of behaviors than the solutions of the corresponding differential equation.

Consider the differential equation

$$(\mathrm{e}_1) \qquad y'''' - y = 0.$$

The general solution of (e_1) is

$$y = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t.$$

Generalizing (e_1) slightly, consider the equation

$$(\mathrm{e}_2) \qquad y'''' - A^4 y = 0,$$

Where A is a positive constant.

The general solution of (e_2) is

$$y = c_1 e^{At} + c_2 e^{-At} + c_3 \cos At + c_4 \sin At.$$

The solution $y_1 = e^{At}$ is called a strongly monotonic solution since $y_1 > 0,\ y_1' > 0,\ y_1'' > 0,\ y_1''' > 0$,

While the solution $y_2 = e^{-At}$ is termed a weakly monotonic solution because

$y_2 > 0,\ y_2' < 0,\ y_2'' > 0,\ y_2''' < 0$.

The solutions $y_3 = \cos At$ and $y_4 = \sin At$ are bounded oscillatory solutions with constant amplitudes. In other works, the solutions y_1 and y_2 have been referred to as strongly increasing functions and strongly decreasing functions, respectively.

A fourth order difference equation having the same characteristic equation as (e_2) is

$$(\mathrm{e}_3) \qquad u_{n+4} - A^4 u_n = 0,\ \ A > 0$$

Whose general solution is

$$u_n = c_1 A^n + c_2 (-A)^n + c_3 A^n \cos \tfrac{n\pi}{2} + c_4 A^n \sin \tfrac{n\pi}{2}.$$

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Clearly, the solutions of (e_3) behave very differently than those of (e_2). Indeed, all solutions of (e_3) converge to 0 when $A < 1$, which is not possible for solutions of (e_2); all solutions of (e_3) are periodic when $A = 1$, another impossibility for solutions of (e_2); and finally, when $A > 1$, all nontrivial solutions of (e_3) are unbounded. Of course both (e_2) and (e_3) have oscillatory and nonoscillatory solutions. But because of the wide disparity in the behavior of solutions, (e_3) is not considered as a "good" discrete analogue of (e_2).

Another possible discrete analogue of (e_2) is the difference equation

$$(e_4) \qquad \Delta^4 u_n - A^4 u_n = 0,\ A > 0$$

Where Δ denotes the forward difference operator, i.e. $\Delta u_n = u_{n+1} - u_n$. However, when $A = 1$, this equation is no longer of order 4. So (e_4) seems to fail as a "good" discrete analogue of (e_2). A search of the literature, see $[1]$, $[3]$, leads one to believe that the equation

$$(1) \qquad \Delta^4 u_n - A^2 u_{n+2} = 0,\ A > 0$$

Is probably the "best" discrete analogue of (e_2). But why? This work attempts to answer this question. For additional studies that compare properties of solutions of differential equations to those of difference equations with constant coefficients, see $[2]$.

II. (1) $\Delta^4 u_n - A^2 u_{n+2} = 0,\ A > 0$

Since (1) has constant coefficients, to solve this equation requires the use of an associated characteristic equation. The characteristic equation of (1) is

$$(t-1)^4 - A^2t^2 = 0 \text{ Or } \left[(t-1)^2 - At\right]\cdot\left[(t-1)^2 + At\right] = 0,$$

which leads to the following pair of quadratic equations

$$(2) \qquad (t-1)^2 - At = 0$$

$$(3) \qquad (t-1)^2 + At = 0.$$

The solutions of (2) are

$$t = \frac{2 + A \pm \sqrt{(2+A)^2 - 4}}{2} = \frac{2 + A \pm \sqrt{A^2 + 4A}}{2}.$$

Let $t_1 = \dfrac{2+A+\sqrt{A^2+4A}}{2}$ and $t_2 = \dfrac{2+A-\sqrt{A^2+4A}}{2}$.

Similarly, the solutions of (3) are

$$t = \frac{2 - A \pm \sqrt{(2-A)^2 - 4}}{2} = \frac{2 - A \pm \sqrt{A^2 - 4A}}{2}.$$

Let $t_3 = \dfrac{2-A+\sqrt{A^2-4A}}{2}$ and $t_4 = \dfrac{2-A-\sqrt{A^2-4A}}{2}$.

III. Types of Solutions

Since $A^2 + 4A > 0$, the first two roots are real and determine, respectively, a strongly monotonic solution and a weakly monotonic solution. To see this:

Since $t_1 = 1 + \dfrac{A + \sqrt{A^2 + 4A}}{2} > 1$, the corresponding solution is strongly monotonic because $u_n = t_1^n > 0,\ \Delta u_n = t_1^n(t_1 - 1) > 0,\ \Delta^2 u_n = t_1^n(t_1-1)^2 > 0,\ \Delta^3 u_n = t_1^n(t_1-1)^3 > 0.$

Since $t_2 = 1 + \dfrac{A - \sqrt{A^2 + 4A}}{2}$, we will show that $0 < t_2 < 1$ and thus the corresponding solution is weakly monotonic because $u_n = t_2^n > 0,\ \Delta u_n = t_2^n(t_2 - 1) < 0,\ \Delta^2 u_n = t_2^n(t_2-1)^2 > 0,\ \Delta^3 u_n = t_2^n(t_2-1)^3 < 0.$

To this end, consider the inequality

$$A^2 < A^2 + 4A < (A+2)^2$$

which implies

$$A < \sqrt{A^2 + 4A} < A + 2.$$

So

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$$0 < -A + \sqrt{A^2 + 4A} < 2$$

which implies

$$-2 < A - \sqrt{A^2 + 4A} < 0$$

which implies

$$-1 < \frac{A - \sqrt{A^2 + 4A}}{2} < 0$$

and therefore

$$0 < 1 + \frac{A - \sqrt{A^2 + 4A}}{2} = t_2 < 1.$$

Now let us consider what the remaining two roots yield. Recall from above that

$$t_3 = \frac{2 - A + \sqrt{A^2 - 4A}}{2} \quad \text{and} \quad t_4 = \frac{2 - A - \sqrt{A^2 - 4A}}{2}.$$

Since $A^2 - 4A = A(4 - A)$ and $A > 0$, the roots are complex conjugates when $0 < A < 4$, repeated when $A = 4$, and real and distinct when $A > 4$.

Case 1: $0 < A < 4$

$$t_3 = \frac{2 - A}{2} + \frac{\sqrt{4A - A^2}}{2} i,$$

$$|t_3|^2 = \left(\frac{2-A}{2}\right)^2 + \left(\frac{\sqrt{4A - A^2}}{2}\right)^2 = \frac{4 - 4A + A^2 + 4A - A^2}{4} = 1.$$

Accordingly, $|t_3| = |t_4| = 1$ for $0 < A < 4$, the corresponding solutions of (1) take the form $u_n = \cos n\theta$ and $v_n = \sin n\theta$, where $\theta = \arg t_3$, and are bounded and oscillatory. Thus we see that (1) is an excellent discrete analogue of (e_2).

Case 2: $A = 4$

$$t_3 = t_4 = \frac{2 - 4}{2} = -1.$$

This leads to a pair of oscillatory solutions, $u_n = (-1)^n$ and $v_n = n(-1)^n$, one bounded and the other unbounded.

Case 3: $A > 4$

$$t_4 = \frac{2 - A - \sqrt{A^2 - 4A}}{2} < \frac{2 - A + \sqrt{A^2 - 4A}}{2} = t_3$$

In fact, $t_4, t_3 < 0$ because

$$2 - A + \sqrt{A^2 - 4A} < 2 - A + \sqrt{A^2 - 4A + 4} = 0.$$

Consequently, the corresponding solutions $u_n = t_3^n$ and $v_n = t_4^n$ are oscillatory. A more careful analysis (see the following claim) reveals that the t_4-solution is an unbounded oscillatory solution and the other oscillatory solution converges to zero. This behavior cannot occur in fourth order linear differential equations with constant coefficients. So Case 2 and Case 3 illustrate why solutions of (1) are very interesting in their own right. However the solutions of (1) have a much richer assortment of behaviors, as we see below.

Claim: $t_4 < -1 < t_3$

Proof: The inequality $4 < A < A + \sqrt{A^2 - 4A}$ implies

$$4 - A - \sqrt{A^2 - 4A} < 0$$

which implies

$$2 - A - \sqrt{A^2 - 4A} < -2$$

and therefore

$$t_4 = \frac{2 - A - \sqrt{A^2 - 4A}}{2} < -1.$$

Also the inequality $4 < A$ leads to the following sequence of inequalities:

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$$-4A+16<0$$
$$A^2-8A+16<A^2-4A$$
$$A-4<\sqrt{A^2-4A}$$
$$-2<2-A+\sqrt{A^2-4A}$$

and therefore

$$-1<\frac{2-A+\sqrt{A^2-4A}}{2}=t_3.$$

Q.E.D.

Thus the t_3-solution oscillates and converges to 0, while the t_4-solution oscillates and is unbounded. Clearly, $A=4$ represents a bifurcation for the system (1), in the sense that the oscillatory solutions transition from solutions with constant amplitudes to solutions with variable amplitudes. Again we emphasize that this type of behavior does not occur with solutions of (e_2).

IV. Root Dynamics

We now examine the roots of (2) and (3) as functions of A. Recall that the roots of (2) are

$$t_1=1+\frac{A+\sqrt{A^2+4A}}{2}>1 \text{ and}$$

$$t_2=\frac{2+A-\sqrt{A^2+4A}}{2},\ 0<t_2<1.$$

Clearly $\lim_{A\to\infty} t_1=\infty$. Moreover, $\lim_{A\to\infty} t_2=0$ because

$$\lim_{A\to\infty}\frac{(2+A)-\sqrt{A^2+4A}}{2}=\lim_{A\to\infty}\frac{(2+A)^2-\left(\sqrt{A^2+4A}\right)^2}{2\left[(2+A)+\sqrt{A^2+4A}\right]}=\lim_{A\to\infty}\frac{4}{4t_1}=0.$$

The toots of (3) $t_3=\dfrac{2-A+\sqrt{A^2-4A}}{2}$ and $t_4=\dfrac{2-A-\sqrt{A^2-4A}}{2}$ require more analysis.

For $0\le A\le 4$:

$\mathrm{Re}(t_3)=\mathrm{Re}(t_4)=\dfrac{2-A}{2}$ and

$$\mathrm{Im}(t_3)=\frac{\sqrt{4A-A^2}}{2}\ge 0,\ \mathrm{Im}(t_4)=-\frac{\sqrt{4A-A^2}}{2}\le 0.$$

As A increases from 0 to 4, the real part decreases from 1 to -1, $\mathrm{Im}(t_3)$ increases from 0 to 1 (when $A=2$) and then decreases back to 0, while $\mathrm{Im}(t_4)$ decreases from 0 to -1 and then increases back to 0. Accordingly, as A increases from 0 to 4, t_3 and t_4 both travel from 1 to -1 around the unit circle in the complex plane, t_3 along the upper half and t_4 along the lower half.

For $A>4$:

$\lim_{A\to\infty} t_4=-\infty$ because

$$t_4=\frac{2-A-\sqrt{A^2-4A}}{2}<\frac{2-A}{2};$$

while $\lim_{A\to\infty} t_3=0$ because

$$t_3=\frac{2-A+\sqrt{A^2-4A}}{2}=\frac{(2-A)^2-\left(\sqrt{A^2-4A}\right)^2}{2\left(2-A-\sqrt{A^2-4A}\right)}=\frac{4}{4t_4}.$$

Summary: As A increases from 0 to ∞, t_1 increases from 1 to ∞, t_2 decreases from 1 to 0, t_3 starts at 1 and travels around the upper half of the unit circle in the complex plane to -1 and then increases to 0 along the negative real axis, t_4 travels from 1 to -1 along the lower half the circle and then decreases to $-\infty$ along the negative real axis.

V. Root Inequalities and Dominance Properties of Solutions

For all values of $A>0$, $0<t_2<|t_3|\le|t_4|<t_1$ and for $A>4,\ t_2<|t_3|<1<|t_4|<t_1$. Most of these inequalities are clear by our previous results. Only two of these still require verification: (a.) $|t_4|<t_1$ and (b.) $t_2<|t_3|$.

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(a.) $|t_4| < t_1$: If $0 < A \le 4$, then $|t_4| = 1 < t_1$. If

$A > 4$, then $t_4 = \dfrac{2 - A - \sqrt{A^2 - 4A}}{2} < 0$

and therefore $|t_4| = \dfrac{A - 2 + \sqrt{A^2 - 4A}}{2}$.

We compare this with $t_1 = \dfrac{2 + A + \sqrt{A^2 + 4A}}{2}$.

Since $A - 2 < 2 + A$ and $A^2 - 4A < A^2 + 4A$, then

$A - 2 + \sqrt{A^2 - 4A} < 2 + A + \sqrt{A^2 + 4A}$.

Accordingly,

$$|t_4| = \frac{A - 2 + \sqrt{A^2 - 4A}}{2} < \frac{2 + A + \sqrt{A^2 + 4A}}{2} = t_1.$$

(b.) $t_2 < |t_3|$: If $0 < A \le 4$, $t_2 < 1 = |t_3|$.

For

$A > 4$, $|t_3| = -t_3 = \dfrac{A - 2 - \sqrt{A^2 - 4A}}{2}$ and $t_2 = \dfrac{2 + A - \sqrt{A^2 + 4A}}{2}$.

We wish to prove

$$\frac{2 + A - \sqrt{A^2 + 4A}}{2} < \frac{A - 2 - \sqrt{A^2 - 4A}}{2}$$

which is equivalent to the following inequalities:

$$2 + A + \sqrt{A^2 - 4A} < A - 2 + \sqrt{A^2 + 4A}$$

$$\sqrt{A^2 - 4A} + 4 < \sqrt{A^2 + 4A}. \qquad (*)$$

Since $\sqrt{A^2 - 4A} < \sqrt{A^2 - 4A + 4} = A - 2$, then

$$\sqrt{A^2 - 4A} + 2 < A$$

$$8\sqrt{A^2 - 4A} + 16 < 8A$$

$$A^2 + 8\sqrt{A^2 - 4A} + 16 < A^2 + 8A$$

$$A^2 - 4A + 8\sqrt{A^2 - 4A} + 16 < A^2 + 4A$$

$$\left(\sqrt{A^2 - 4A} + 4\right)^2 < A^2 + 4A$$

$$\sqrt{A^2 - 4A} + 4 < \sqrt{A^2 + 4A}.$$

Since this is inequality $(*)$ above, we have established $t_2 < |t_3|$.

Let $T_3(n)$ and $T_4(n)$ be independent oscillatory solutions of (1) derived from t_3 and t_4. If $u_n = C_1 t_1^n + C_2 t_2^n + C_3 T_3(n) + C_4 T_4(n)$ is a nontrivial solution of (1), then

(a) u_n is strongly monotonic if and only if $C_1 \ne 0$;

(b) u_n is oscillatory if and only if $C_1 = 0$ and $C_3^2 + C_4^2 \ne 0$;

(c) u_n is weakly monotonic otherwise.

Note: These conclusions remain true when $A = 4$ since then $t_1 > 1$, $t_2 < 1$ and $t_3 = t_4 = -1$; thus if $C_1 \ne 0$, then $u_n = C_1 t_1^n + C_2 t_2^n + C_3(-1)^n + C_4 n(-1)^n$ is strongly monotonic because $\lim_{t \to \infty} t_1^n - n = \infty$. Moreover, when $A > 4$, u_n is unbounded if either $C_1 \ne 0$ or $C_4 \ne 0$, otherwise u_n converges to 0.

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